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SHOCK-FREE CONICAL COMPRESSION AND EXPANSION OF A GAS†

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Solutions of two-dimensional unsteady self-similar problems of the unlimited shock-free compression and expansion of an ideal gas into a vacuum when the gas is at rest at the initial instant of time inside a prism and cone-shaped bodies at constant density and pressure are constructed. The flow fields are partially constructed using classes of accurate solutions of the non-linear equation for the velocity potential, and partially by numerical calculations, in particular, by the method of characteristics. The features of the formulation of the boundary-value problems for conical unsteady flows are investigated. Approximate laws of the control of the motion of compressing pistons are constructed analytically. The degrees of cumulation of energy and density are obtained and it is shown that the non-uniform compression processes described are more favourable energy-wise than the process of spherical compression for obtaining local superhigh densities of a material. The flow fronts with points of discontinuity are constructed for problems of flow into a vacuum from a cone.

Non-uniform processes of unlimited shock-free compression of ideal polytropic gases which, at the initial instant of time, are inside a prism, tetrahedra and cone-shaped bodies of special shape were constructed previously in [1–3]. It was shown that high-velocity gas jets with a density that increases without limit are formed in these processes. The input energy required to achieve very high local densities of the material are much less than in the case of one-dimensional spherical processes of shock-free compression, used, in particular, to initiate laser thermonuclear fusion [4, 5].

The considerations in [1–3] were based on exact two-dimensional and three-dimensional self-similar solutions of the equations of gas dynamics, which were only constructed for an adiabatic index γ and initial geometrical parameters of the compressed volumes of gas which were matched to one another in a special way (the matched case). It was for such solutions, which belong to classes of motion with uniform deformation [6, 7], that the laws of control of the motion of mobile compressing pistons were constructed, leading to unlimited compression.

At the same time, estimates of the limiting degrees of cumulation of the density and energy [2], and also estimates of the parameters of the corresponding economic processes of compression require a consideration of more general classes of solutions. This paper is devoted to a more detailed analysis of one of these classes of accurate solutions [2], which, in the general case, already possesses the property of motions with uniform deformation. In addition to problems of unlimited plane and axisymmetrical shock-free compression, using this class of solutions we also solve the problem of the flow of a gas into a vacuum from an unbounded cone.

1. Suppose that at the initial instant of time $t = 0$ a polytropic gas with the equation of state

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$$p = a^2\rho^\gamma$$

(p is the pressure, ρ is the density and $a^2 = \text{const}$) and $p = p_0 = \text{const}$, $\rho = \rho_0 = \text{const}$, $c_0 = 1$ (c_0 is the initial velocity of sound $\gamma p_0 = \rho_0$) is at rest inside a prism with cross-section AOB (the plane case) or a solid of rotation with generatrix ABO (z is the axis of rotation, and $|OB| = 1$ (Fig. 1)). The line ABO corresponds to the initial position of a movable piston S_t, the law of motion of which must be determined so that, during adiabatic compression with constant entropy, the entire gas at the instant of time $t = 1$ is focused at the point 0. The instant of time $t = 1$ corresponds to the time that a sonic perturbation traverses the section OB (at the instant $t < 1$ the line GH), which is separated at the initial instant of time from the line AB—part of the piston S_0. The straight line OB can then serve both as a fixed impenetrable wall during the whole compression process, and correspond to the initial position of the movable part of the piston S_t so that at the instant of time t the line $DEF'H$ corresponds to it, while for the fixed wall OB the line DEF corresponds to it.

Only weak discontinuities can be present in shock-free flows of gas. Hence, the perturbed motion will be potential. The equation for the velocity potential $\Phi(t, r, z)$ has the form

$$\Phi_{tt} + 2\Phi_r\Phi_{rt} + 2\Phi_z\Phi_{zt} + 2\Phi_r\Phi_z\Phi_{rz} + \Phi_r^2\Phi_{rr} + \Phi_z^2\Phi_{zz} - \theta(\Phi_{rr} + \Phi_{zz} + Nr^{-1}\Phi_r) = 0$$
$$\theta = c^2 = (\gamma - 1)(K - \Phi_t - \tfrac{1}{2}\Phi_r^2 - \tfrac{1}{2}\Phi_z^2)$$

where c is the velocity of sound, $K = \text{const}$, $u_r = \Phi_r$, $u_z = \Phi_z$, u_r and u_z are the components of the velocity vector, and $N = 0$ corresponds to the plane case, and $N = 1$ corresponds to the axisymmetrical case.

We will construct solutions of the problem of compression in the region $DGHFE$ in the class of conical self-similar flows of gas with independent variables

$$\xi = z/\tau, \quad \eta = r/\tau, \quad \tau = t - 1, \quad t \in [0, 1] \tag{1.1}$$

Assuming $\Phi = Kt - \tau\Psi(\xi, \eta)$ the equation of conical flows can be written in the form

$$\begin{aligned}&(\Psi_\xi + \xi)^2\Psi_{\xi\xi} + 2(\Psi_\xi + \xi)(\Psi_\eta + \eta)\Psi_{\xi\eta} + (\Psi_\eta + \eta)^2\Psi_{\eta\eta} - \\ &-(\gamma - 1)(\Psi - \xi\Psi_\xi - \eta\Psi_\eta - \tfrac{1}{2}\Psi_\xi^2 - \tfrac{1}{2}\Psi_\eta^2)(\Psi_{\xi\xi} + \Psi_{\eta\eta} + N\eta^{-1}\Psi_\eta) = 0\end{aligned} \tag{1.2}$$

We will introduce the new unknown function

$$\Gamma = \Psi + \tfrac{1}{2}(\xi^2 + \eta^2)$$

From (1.2) we obtain the following equation for Γ

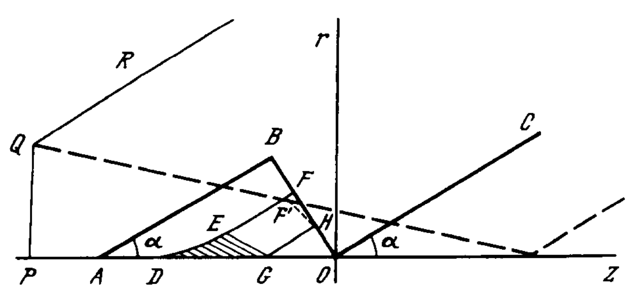

Fig. 1.

$$\Gamma_\xi^2(\Gamma_{\xi\xi}-1)+2\Gamma_\xi\Gamma_\eta\Gamma_{\xi\eta}+\Gamma_\eta^2(\Gamma_{\eta\eta}-1)-$$
$$-(\gamma-1)(\Gamma-\tfrac{1}{2}\Gamma_\xi^2-\tfrac{1}{2}\Gamma_\eta^2)(\Gamma_{\xi\xi}+\Gamma_{\eta\eta}-N-2+N\eta^{-1}\Gamma_\eta)=0 \tag{1.3}$$

which is invariant to a shift with respect to the variable ξ. This will be used when constructing the solutions.

The type of Eq. (1.3) is determined by the law of the discriminant D

$$D=\theta(\Gamma_\xi^2+\Gamma_\eta^2-\theta),\quad \theta=(\gamma-1)(\Gamma-\tfrac{1}{2}\Gamma_\xi^2-\tfrac{1}{2}\Gamma_\eta^2)\geqslant 0$$

In the case of a weak discontinuity of GH we have

$$\Psi_\xi=\Psi_\eta=0,\quad \theta=1 \tag{1.4}$$

while at the point H we have $\xi^2+\eta^2=1$. Hence, for compression waves ($\theta\geqslant 1$) over the whole region considered, apart from the point H, we will have $D>0$, and Eq. (1.3) is of the hyperbolic type. At the point H we have $D=0$, and the equation becomes degenerate.

For the case of unbounded compression, when OB is a fixed wall, in the plane of the self-similar variables ξ, η the unbounded region $B'H'G'A'$ corresponds to the region of flow (Fig. 2). The function θ must increase without limit as $\xi^2+\eta^2\to\infty$. In addition to relations (1.4) the function θ must satisfy the conditions for no flow to occur on the movable boundaries and the line $\eta=0$. These conditions and conditions (1.4) are insufficient to construct non-trivial solutions of the problem of the shock-free compression of a gas and to determine the laws of motion of the surfaces controlling the compression.

The situation is a non-standard one and is difficult both for finding solutions by analytic construction and by constructing numerical methods of calculating these compression processes even when high-power computers are available. Note that the equation of conical unsteady flows (1.2) when $N=1$ has a particularly complex structure, which differs from the structure of the equation for the velocity potential in the case of three-dimensional steady conical flows of gas [8]. Although a number of features of the equation are common (the variability of the type of equation in general, and the conservation of the flow parameters along the rays), the formulation of the problems and the properties of the solutions are, as a rule, quite different.

One of the ways of solving the problem rests [2] on the analytic construction of the flow in the region DEG, where the most powerful cumulative jet is formed, and the calculation, by highly accurate difference methods, of the flow in the remaining parts of the region of the perturbed motion.

We will consider one form of this approach in more detail.

After changing in the ξ, η plane to polar coordinates $\xi=\mu\cos\lambda-\xi_0$, $\eta=\mu\sin\lambda$, where $\xi_0=$ const, Eq. (1.3) takes the form

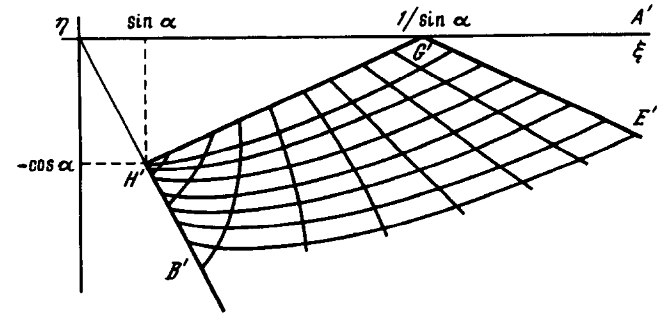

Fig. 2.

$$\Gamma_\mu^2\Gamma_{\mu\mu} + 2\mu^{-2}\Gamma_\mu\Gamma_\lambda\Gamma_{\mu\lambda} + \mu^{-4}\Gamma_\lambda^2\Gamma_{\lambda\lambda} - \Gamma_\mu^2 - \mu^{-2}\Gamma_\lambda^2 - \mu^{-3}\Gamma_\mu\Gamma_\lambda^2 -$$
$$-(\gamma-1)(\Gamma - \tfrac{1}{2}\Gamma_\mu^2 - \tfrac{1}{2}\mu^{-2}\Gamma_\lambda^2)\times$$
$$\times[\Gamma_{\mu\mu} + \mu^{-2}\Gamma_{\lambda\lambda} + \mu^{-1}\Gamma_\mu - N - 2 + N(\mu^{-1}\Gamma_\mu + \mu^{-2}\Gamma_\lambda \operatorname{ctg}\lambda)] = 0 \tag{1.5}$$

We will construct a class of accurate solutions of (1.5) in the form [2]

$$\Gamma = \mu^2 A(\lambda) \tag{1.6}$$

where $A(\lambda)$ satisfies the equation

$$A'^2A'' + 8A^3 + 6AA'^2 - 4A^2 - A'^2 -$$
$$-(\gamma-1)(A - 2A^2 - \tfrac{1}{2}A'^2)[2(2+N)A - 2 - N + A'' + NA'\operatorname{ctg}\lambda] = 0 \tag{1.7}$$

The condition $\Psi_\eta = 0$ must be satisfied on the axis $\eta = 0$, which leads to the relation

$$A'(0) = 0 \tag{1.8}$$

By specifying the second initial condition when $\lambda = 0$, $A(0) = a$ in the case when $0 < a < \tfrac{1}{2}$, we can obtain the law of motion of the point A on the piston which controls the compression along the z axis. In fact, from the relation

$$\Gamma_\xi = \Gamma_\mu \cos\lambda - \mu^{-1}\sin\lambda\Gamma_\lambda$$

with $\lambda = 0$ we obtain $\Psi_\xi + \xi = 2(\xi + \xi_0)a$. Assuming $\Psi_\xi = -\Phi_z = -dz/d\tau$ on the piston at the point A, we obtain the following equation for $z(\tau)$

$$dz/d\tau = (1-2a)z/\tau - 2\xi_0 a$$

whence we obtain

$$z = C(-\tau)^{1-2a} - \xi_0\tau, \quad C = -(\sin\alpha)^{-1} \tag{1.9}$$

It follows from (1.9) that the degree of cumulation of the velocity n_u as $\tau \to 0$ is $2a$ ($|\mathbf{u}| = O((-\tau)^{-n_u}$. For the velocity of sound also the degree of cumulation $n_c = 2a$. The quantity ξ_0 can be found from the condition that at the point G' (Fig. 2) the relations $c = 1$, $\Psi_\xi = \Psi_\xi = 0$ are satisfied.

This leads to the following representations

$$\xi_0 = (\gamma-1)^{-1/2}a^{-1/2}(1-2a)^{-1/2} - (\sin\alpha)^{-1}$$

Equation (1.7) can be written in the form

$$A'' = PQ^{-1} \tag{1.10}$$
$$P = (\gamma-1)(A - 2A^2 - \tfrac{1}{2}A'^2)[2(2+N)A - 2 - N + NA'\operatorname{ctg}\lambda] - 8A^3 - 6AA'^2 + 4A^2 + A'^2$$
$$\theta = A'^2 - (\gamma-1)(A - 2A^2 - \tfrac{1}{2}A'^2)$$

The initial conditions at the point $\lambda = 0$ when $0 < a < \tfrac{1}{2}$ enable us, with $N = 0$, to guarantee the existence of a solution of the Cauchy problem at least in the neighbourhood of $\lambda = 0$. When $N = 1$ the point $\lambda = 0$ is singular. Nevertheless, as will be shown later, a solution can also be constructed.

An important problem when constructing solutions with unlimited cumulation is the question of the form and behaviour of the characteristics of Eq. (1.2). The flow as a whole will be constructed from solutions with different analytic structure in subregions of the ξ, η plane, separated by certain characteristics (lines of weak discontinuities of the solutions of Eq. (1.2)).

The equations of the characteristics (1.2) have the form

$$\frac{d\eta}{d\xi}=\frac{-\Gamma_\xi\Gamma_\eta\pm\sqrt{\theta(\Gamma_\xi^2+\Gamma_\eta^2-\theta)}}{\theta-\Gamma_\xi^2}$$

For the class of solutions (1.6) they can be converted to the form

$$\frac{d\xi}{d\lambda}=-\frac{\mu L(\lambda)}{\Delta(\lambda)},\quad \frac{d\eta}{d\lambda}=-\frac{\mu M_\pm(\lambda)}{\Delta(\lambda)} \tag{1.11}$$

$$\Delta(\lambda)=L(\lambda)\sin\lambda-M_\pm(\lambda)\cos\lambda$$

$$L(\lambda)=(\gamma-1)(A-2A^2-\tfrac{1}{2}A'^2)-(2A\cos\lambda-A'\sin\lambda)^2$$

$$M_\pm(\lambda)=\tfrac{1}{2}(A'^2-4A^2)\sin 2\lambda-2AA'\cos 2\lambda\pm$$

$$\pm\{(\gamma-1)(A-2A^2-\tfrac{1}{2}A'^2)[2(\gamma+1)A^2+(\gamma-1)(A'^2/2-A)]\}^{1/2}$$

From (1.11) for the function $\mu=\mu(\lambda)$ along the characteristics we obtain the equations

$$\frac{d\ln|\mu|}{d\lambda}=-\frac{1}{2}\frac{L(\lambda)\cos\lambda+M_\pm(\lambda)\sin\lambda}{L(\lambda)\sin\lambda-M_\pm(\lambda)\cos\lambda} \tag{1.12}$$

2. We will consider the problem of constructing solutions of the form (1.6) in the region DEG (Fig. 1), corresponding to the sector $A'E'G'$ (Fig. 2). The exact solutions of Eq. (1.7) with a special choice of a was used in [1, 2]. They have the form

$$A=\chi\frac{\gamma-1}{2(\gamma+1)}(1+\cos 2\lambda),\quad \chi=\begin{cases}1, & N=0, \quad \gamma<3\\ 3/2, & N=1, \quad \gamma<2\end{cases} \tag{2.1}$$

The plane solution for $N=0$, $\gamma<3$ was constructed for the first time in [9] and was used to solve the problem of the flow of gas into a vacuum along a sloping wall.

When considering problems of compression we will assume that $\lambda\leqslant 0$, $\mu\leqslant 0$, in which case $\xi\geqslant 0$, $\eta\leqslant 0$. Note that the even-continued function $A(\lambda)$ will also be a solution of Eq. (1.7). It turned out that curve (2.1) of the type $N=0$, $\gamma<3$ passes through a singular point of Eq. (1.7) of the saddle type, in which

$$A=\frac{\gamma-1}{4},\quad A'=\frac{\gamma-1}{2}\sqrt{\frac{3-\gamma}{\gamma+1}},\quad \lambda=\lambda_s^{(0)}=-\frac{1}{2}\arccos\frac{\gamma-1}{2} \tag{2.2}$$

Curve (2.1) when $N=1$, $\gamma<2$ passes through two singular points: $\lambda=0$ $(A'(0)=0)$ and the point at which

$$A=\frac{\gamma-1}{2},\quad A'=(\gamma-1)\sqrt{\frac{2-\gamma}{\gamma+1}},\quad \lambda=\lambda_s^{(1)}=-\frac{1}{2}\arccos\frac{2\gamma-1}{3} \tag{2.3}$$

For the compression problem the region in which the solution is defined is $\lambda\in(\lambda_s^{(j)},0)$ $(j=0, 1)$. The characteristics in the region $A'G'E'$ of the ξ, η plane are straight lines and depart to infinity.

It is difficult to make an analytic investigation of the fields of the integral curves of Eq. (1.7), particularly when $N=1$, due to the presence of a mobile singularity (the non-autonomous problem). When $N=0$, although the order of Eq. (1.7) is reduced and, as a result, one obtains an autonomous Abel equation of the second kind, it is also difficult to prove the fact that for any $0<a<1/2$ the integral curve passes through the saddle (2.2) with some other λ_s. Hence, the fact that integral curves of Eq. (1.7) exist, which connect two singular points with arbitrary $0<a<1/2$ for $N=1$, and the above-mentioned fact for $N=0$, is established by a highly accurate numerical integration of Eq. (1.7) using several methods employing analytic expansions in the neighbourhood of the singular points.

We will first establish that if $P=Q=0$ in (1.10) for some λ (the unmatched general case), the denominator in (1.12) will also vanish, and the characteristic corresponding to M_+ departs to infinity in the ξ, η plane.

In fact, using the relation

$$A'^2=2(\gamma-1)(\gamma+1)^{-1}A(1-2A) \tag{2.4}$$

which follows from the equality $Q=0$ for L and M_+ in (1.11) we obtain the representations

$$\begin{gathered} L=R\cos\lambda, \quad M_+=R\sin\lambda \\ R=(A'^2-4A^2)\cos\lambda+4AA'\sin\lambda \end{gathered} \tag{2.5}$$

The above assertion then follows from (2.5). The numerator in (1.12) in this case, generally speaking, will not be equal to zero. Note that the point of unlimited compression O, by virtue of (1.9), corresponds to infinitely distant points of the sector $B'H'G'A'$ in the ξ, η plane.

In the plane case, the point $\lambda=0$ is a singular point. Expanding the indeterminacy in the expression for $A'\mathrm{ctg}\lambda$ with $N=1$, we obtain

$$A''(0)=3/2-2a\gamma-a \tag{2.6}$$

For the axisymmetric case at a singular point of the saddle type with $\lambda=\lambda_s$, when P and Q in (1.10) vanish, we obtain

$$\begin{aligned} A(\lambda_s)&=\frac{(\gamma-1)^3+(\gamma^2-1)[4(\gamma+1)\mathrm{tg}\lambda_s-\sqrt{(\gamma-1)^2+8(3-\gamma)\mathrm{tg}^2\lambda_s}\,]}{4[(\gamma-1)^3+(\gamma+1)^3\mathrm{tg}^2\lambda_s]} \\ A'(\lambda_s)&=2\,\mathrm{tg}\lambda_s[1-(\gamma+1)(\gamma-1)^{-1}A(\lambda_s)] \end{aligned} \tag{2.7}$$

For $A''(\lambda_s)$, passing to the limit in (1.10) as $\lambda\to\lambda_s$ we obtain the following quadratic equation

$$\begin{aligned} &r_2A''^2(\lambda_s)+r_1A'(\lambda_s)+r_0=0 \\ &r_0=A'^2(\lambda_s)(2-1/\cos^2\lambda_s)+8A(\lambda_s)-24A^2(\lambda_s)-2A'^2(\lambda_s) \\ &r_1=(\gamma-1)[1-4A(\lambda_s)]+2[A(\lambda_s)(\mathrm{ctg}\lambda_s-2)-1]+A'(\lambda_s)\mathrm{ctg}\lambda_s-2-4A(\lambda_s),\ r_2=-\gamma-1 \end{aligned} \tag{2.8}$$

For the plane case r_k in (2.8) are independent of λ_s and have the form

$$r_0=2(4A-12A^2-A'^2), \quad r_1=\gamma(1-4A)-3, \quad r_2=-\gamma-1$$

where A and A' are taken from (2.3).

Representations (2.6)–(2.8) enable us to establish an approximate form of the analytic expansions for the function $A(\lambda)$ in the neighbourhood of singular points, obtain the slopes of the separatrices and integrate (1.10) using these expansions both from $\lambda=0$ and from $\lambda\to\lambda_s$. It turned out that the problem of numerical integration encounters a number of difficulties:

instability of the calculation connected with the direction of integration, and high sensitivity to the choice of the value of the steps. Nevertheless, by using four different numerical methods, we were able, using numerical methods and a very fine integration step ($\sim 10^{-4}$), to establish the existence of integral curves connecting two singular points for the axisymmetric case, and a singular point of the saddle type with the point $\lambda = 0$, $A(0) = a$ in the plane case.

In Fig. 3 we show curves of $A(\lambda)$ calculated from $\lambda = 0$ to the corresponding λ_{s_i} for $\gamma = \frac{5}{3}$. Curves 1 ($N = 1$) and 2 $N = 0$ correspond to the matched cases (2.1), and curves 3–6 correspond to the unmatched cases with $a_i = 0.3$, 0.4, 0.48 and 0.495. Curves 1 and 2 were obtained for $\lambda \in [-\pi/2, 0]$. In Fig. 4 we show the characteristics $G'E'$ corresponding to the calculated $A(\lambda)$ (Fig. 3) ($\alpha_1 = 19°.5; \alpha_2 = 35°.2; \alpha_3 = 28°.1; \alpha_4 = 16°.7; \alpha_5 = 6°.8; \alpha_6 = 3°.3$). To keep the figure compact the initial point $\eta = 0$, $\xi = (\sin\alpha)^{-1}$ of all the characteristics is shifted along the axis to the origin of coordinates in Fig. 4.

3. The further calculation of the flow fields for the plane and axisymmetric cases can be carried out differently. In the plane case, using the theorem that travelling waves of different ranks touch along weak discontinuities [10, 11], we can construct the solution in the section $B'H'G'E'$ (Fig. 2) from the class of two-dimensional self-similar simple waves, which continuously approach a solution of the form (1.6) in the sector $E'G'A'$ along the characteristic $G'E'$. Along $G'E'$ we put

$$u_z = f_1(\kappa), \quad u_r = f_2(\kappa), \quad \kappa = 2c/(\gamma - 1)$$

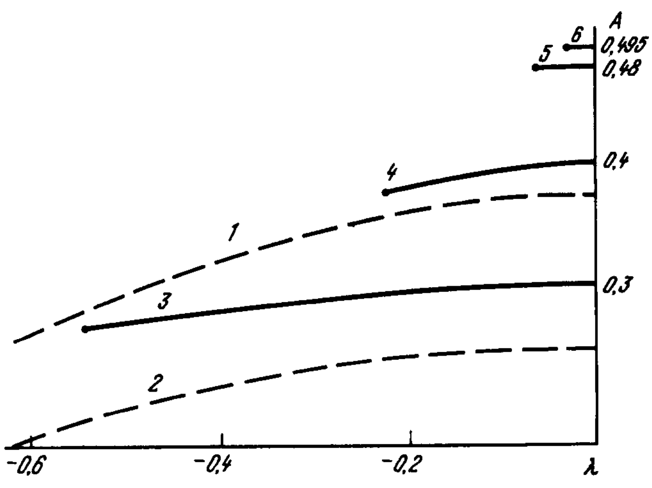

Fig. 3.

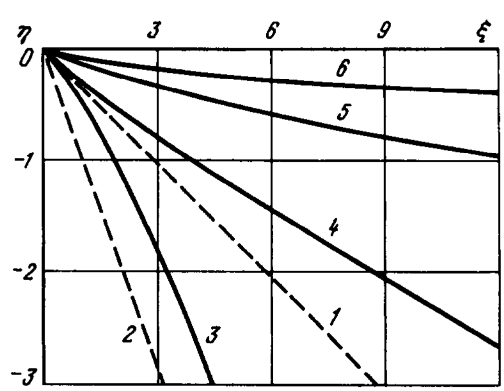

Fig. 4.

where the functions f_1 and f_2 are found after integrating the equations of the characteristics (1.11) and satisfy the relations [11]

$$f_1'^2 + f_2'^2 = 1 \tag{3.1}$$

$$f_1\xi + f_2'\eta - [(\gamma-1)\kappa/2 + f_1 f_1' + f_2 f_2'] = 0$$

The second equation of (3.1), after finding f_1 and f_2, defines the field of the rectilinear characteristics in the sector $B'H'G'E'$, along which the value of the velocity of sound and the components of the velocity vector stay constant. At the point G'

$$f_1(g) = f_2(g) = 0, \quad f_1'(g) = \sin\alpha, \quad f_2'(g) = \cos\alpha, \quad g = 2/(\gamma-1)$$

Finally, here, apart from matched α and $\gamma(\mathrm{tg}^2\alpha = (3-\gamma)/(\gamma+1)$ [1]), it is impossible, generally speaking, to satisfy the no-flow condition on the rectilinear wall OB (Fig. 1) and we need to construct a mobile surface $DEF'H$, by integrating over the constructed field of the velocities u_z and u_r the equations of the characteristics

$$dz/dt = u_z, \quad dr/dt = u_r \tag{3.2}$$

with initial data on the curve ABH. Then, the mobile surface S_t, which is given by the equation $F(t, r, z) = 0$ and satisfies the non-condition

$$F_t + u_z F_z + u_r F_r = 0 \tag{3.3}$$

can be obtained at each fixed instant of time from the characteristics (3.2) by fixing their position at the instant t [12].

There are no classes of simple waves in the axisymmetrical case and the flow in the sector $B'H'G'E'$ corresponds to the general type of solution. It can be constructed numerically by the method of characteristics by solving the Gurs problem with known data on the characteristics $H'G'$ and $G'E'$. Finally, we have to overcome two difficulties here connected with the unbounded nature of the region of integration, the considerable rotation of the characteristics, and the stability of the calculation.

In Figs 5 and 6 we show fragments of the field of the characteristics and the field of the velocity vectors in the plane of self-similar variables for $\gamma = 5/3$ and $\alpha = 19°.5$ (the matched case $\mathrm{tg}^2\alpha = (2-\gamma)/(\gamma+1)$). In Fig. 2 we show a fragment of the field of the characteristics for $\gamma = 1.4$ and $\alpha = 26°.5$. The calculations were carried out up to values of $|\xi|$, $|\eta| \sim 10^3$. It can be seen (Fig. 5) that the velocity vector is very close in direction to the ray OB (the quantity $(\mathbf{u}\cdot\mathbf{n})/|\mathbf{u}| \sim 10^{-2}$, where $\mathbf{u}$ is the velocity vector and $\mathbf{n}$ is the unit vector of the normal to OB), so that we can assume approximately that OB is a fixed unpenetrable wall.

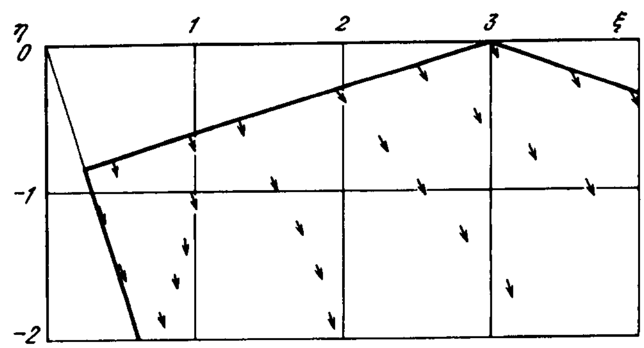

Fig. 5.

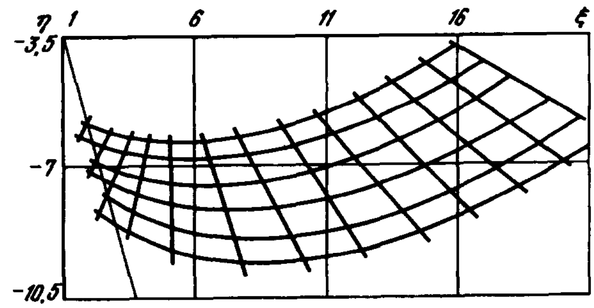

Fig. 6.

We will consider the problem of finding an approximate analytic law of motion of the piston S_t. One version of the approximate form of the law $F(t, r, z)=0$ was derived in [2]. We will construct a more accurate law of motion of S_t. To do this we will first consider, for large $|\xi|$ and $|\eta|$ (in the region of the instant of focusing), the motion of the particles of the gas on the characteristic $G'E'$, defined by the equation

$$\eta=-\xi \operatorname{tg}\alpha+1/\cos\alpha \tag{3.4}$$

Close to the instant of focusing, using (2.4) with $\lambda \sim \lambda_s$ we obtain

$$|\mathbf{u}| \sim \mu q, \quad q=[(1-2A(\lambda_s))(1-4(\gamma+1)^{-1}A(\lambda_s))]^{1/2}$$

Then, assuming that the velocity of the piston on S_t in the direction of the characteristic (3.4) for large μ^2 is identical with $|\mathbf{u}|$, we obtain for $R=\sqrt{(r^2+z^2)}$ in this direction a first-order equation, whence we have

$$R=B(-\tau)^q, \quad B=\text{const} \tag{3.5}$$

For the case of matched α and γ

$$q=q_s=[(2-\gamma)(3-\gamma)/(\gamma+1)]^{1/2} \tag{3.6}$$

Note that the following inequalities are satisfied when $1<\gamma<2$

$$\frac{2}{\gamma+1}>\frac{1}{\gamma}>q_s>q_0=2\frac{2-\gamma}{\gamma+1}$$

Here $2/(\gamma+1)$ and $1/\gamma$ correspond to the laws of motion of plane and cylindrical pistons in the Rayleigh–Hugoniot problem [5], where q_0 is the law of motion (1.8) for matched α and γ. Hence, the degree of cumulation is a maximum on the axis $r=0$.

In order to obtain the law of motion of the piston DE (Fig. 1), i.e. the solution of Eq. (3.3), we need to know the equation of motion of the point E. The hypothesis that the cumulation along the characteristic $G'E'$ is cylindrically uniform was used in [2] to obtain the law of this motion. We will use the refined relation (3.5) and we will assume that the point E moves along $G'E'$ as given by

$$r=D[(-\tau)^{q_s}+\tau], \quad D=\sqrt{3}/[2(\gamma+1)^{1/2}(1-q_s)] \tag{3.7}$$

The integrals of Eqs (3.2) for the matched case considered have the form

$$\frac{r}{\tau} = C_1, \quad (-\tau)^{-2(2-\gamma)/(\gamma+1)}\left(z + \frac{2\sqrt{2-\gamma}}{\sqrt{3}(\gamma-1)}\tau\right) = C_2 \tag{3.8}$$

where C_1 and C_2 are arbitrary constants.

The general solution of Eqs (3.3) can be written in the form $C_2 = \chi(C_1)$, where χ is an arbitrary function, and instead of C_1 and C_2 we substitute their expressions from (3.8). We can obtain the function χ from the requirement that the integral surface of Eq. (3.3) contains the curve (3.7). We finally obtain the following law of motion of the part DE of the piston

$$z = -\sqrt{\frac{\gamma+1}{2-\gamma}}\left(1 - \frac{r}{\tau D}\right)^n\left(-\frac{r}{\tau} + \sqrt{\frac{\gamma+1}{3(\gamma-1)^2}}\right)(-\tau)^m - \frac{2\sqrt{2-\gamma}}{\sqrt{3}(\gamma-1)}\tau = \Lambda(r,\tau)$$

$$n = \frac{3(\gamma-1)}{(\gamma+1)(q_s-1)}, \quad m = 2\frac{2-\gamma}{\gamma+1}, \quad z \leqslant \sqrt{\frac{\gamma+1}{2-\gamma}}r + \sqrt{\frac{3}{2-\gamma}}\tau \tag{3.9}$$

Calculations showed that the part EF of the piston S_t is extremely close in form to a conical surface, while the law of motion of the point F along the wall OB is close to a spherical uniform law of motion of the piston, compressing a sphere of unit radius without limit.

Hence, we have constructed an analytically approximate law of motion of the controlling piston S_t for matched α and γ. We will estimate the order of magnitude of the energy $E(\tau)$ when $\tau \sim 0$, required for unlimited compression

$$E(\tau) = 2\pi\int_{-1}^{\tau}\int_{DF(\tau)} p(\tau)u_n r\,dl\,d\tau = 2\pi a^2\int_{-1}^{\tau}\int_{0}^{r(\tau)} c^{2\gamma/(\gamma-1)}(u_z - u_r\Lambda_r)r\,dr\,d\tau \tag{3.10}$$

Here $p(\tau)$ is the value pf the pressure on S_t, u_n is the value of the component of the velocity vector normal to the surface S_t, dl is an element of length of the arc of the curve DF and $r(\tau)$ is the radius of the point F. To estimate the order of increase of $E(\tau)$ it is sufficient to assume in (3.10) that $r(\tau)$ is found from (3.7), since the part of the piston DE makes the main contribution to the work of the piston S_t.

For matched α and γ with $N = 1$, we will use the following expressions obtained from (2.1)

$$c = \sqrt{3}\frac{\gamma-1}{\gamma+1}\sqrt{2-\gamma}(\xi+\xi_0), \quad \xi_0 = \frac{2\sqrt{2-\gamma}}{\sqrt{3}(\gamma-1)}$$

$$u_z = 2\frac{2-\gamma}{\gamma+1}\xi + 3\frac{\gamma-1}{\gamma+1}\xi_0, \quad u_r = \eta$$

in which on the piston z is a function of r and τ (3.9). Then, after calculating the principal term of the asymptotic form of $E(\tau)$ we obtain

$$E(\tau) = O((-\tau)^{-6(\gamma-1)/(\gamma+1)})$$

We will introduce the quantity $\nu = E(\tau)\rho_m^{-1}(\tau)$, where ρ_m is the maximum density of the gas at the instant τ and $E(\tau)$ is the integral energy required to obtain this compression. For the case of spherical compression [5] $\nu = \nu_s \sim (\tau)^{6(2-\gamma)/(3\gamma-1)}$, and for the conical compression considered $\nu = \nu_c \sim (-\tau)^{6(2-\gamma)/(\gamma+1)}$. If $\gamma < 2$, the quantities ν_s and ν_c will also approach zero as $\tau \to 0$ and the following relation holds

$$\nu_c \sim \nu_s^{(3\gamma-1)/(\gamma+1)} \tag{3.11}$$

The quantity ν represents how economical the compression process is in obtaining high local densities of the material from the point of view of energy costs.

In (3.11) $v_c = o(v_s)$, and hence the process of conical cumulation considered is more economical energy-wise than spherical compression, and moreover give a high degree of cumulation of all the quantities.

The supercumulation effect found in the ideal-gas approximation can play an important role, although, probably, it will weaken the consideration of the actual equations of state, the thermal conductivity, the viscosity, and the radiation at higher temperatures. A large part of the energy E in the process considered goes to increasing the internal energy, in which case the harmful premature considerable overheating of the gas, characteristic for compression using shockwaves, does not occur.

4. The class of solutions (1.6) can be applied not only to problems of high compression, they can also be used to solve problems of the flow of a gas into a vacuum from infinite cones.

Suppose the gas with the same initial parameters as in Section 1 at the instant of time $t=0$ is inside an infinite cone with the vertex at the origin of coordinates and with a semi-aperture angle α (Fig. 1). The side surface of the cone at $t=0$ simultaneously disintegrates and the gas begins to flow into the vacuum. The plane version of this problem was solved in [9].

We will assume that the conditions on the characteristics $H'G'$ and $G'E'$ change places and conditions (1.4) are satisfied along $G'E'$ and representation (2.1) holds along $H'G'$ with $N=1$ and $\lambda \geqslant 0$. (The function $A(\lambda)$ is continued evenly.) Since the decay of the discontinuity occurs at $t=0$ in this problem, instead of ξ and η from (1.1) we must put

$$\xi = z/t, \quad \eta = r/t, \quad t \in (0, \infty)$$

For $\theta = c^2$ the following representation holds

$$\theta = (\gamma-1)\mu^2(A - 2A^2 - A'^2/2)$$

from which it follows that when $\lambda = \pi/2$ and $\xi = -\xi_0 = -2\sqrt{(2-\gamma)}/[\sqrt{(3)}(\gamma-1)]$ the function θ vanishes, which corresponds to the beginning of the vacuum zone. To solve this problem we use the function $A(\lambda)$, defined for $\gamma \in [0, \pi/2]$, i.e. the entire separatrice which passes through the saddle at $\lambda = \frac{1}{2}\arccos[(2\gamma-1)/3]$ (Fig. 3).

Thus suppose solution (2.1) holds in the region $WG'P$ (Fig. 7). Then, we need to solve the Gurs problem in the region $R'WG'F'$ with data on the characteristics WG' and $G'E'$. Again, as in Section 3, when $N=1$ this solution will be a general type of solution. The use of the method of characteristics has a specific feature in that all the characteristics of the dual family emerging from points of the straight line $G'E'$ arrive at a certain neighbourhood of the point W. This fact, although it has also been obtained by numerical calculations, is not accidental.

In the plane case for matched α and γ the solution in the region $TG''UQ$ will be a simple centred Riemann wave. The behaviour of the characteristics in this region can easily be investigated analytically. The equations of the family of characteristics emerging from $G''T$, in the system of coordinates ξ', η', obtained from the initial system of coordinates by rotation by an angle α around the origin of coordinates in a clockwise direction, have the form

$$\xi' = \sqrt{\frac{\gamma+1}{3-\gamma}}\left[\left(\frac{\gamma-1}{\gamma+1}\right)^2\left(\eta' - \frac{2}{\gamma-1}\right)^2 + C_1\left(\eta' - \frac{2}{\gamma-1}\right)^{(\gamma+1)/(\gamma-1)}\right]^{1/2}, \quad C_1 = \text{const}$$

At the point U of the discontinuity of the flow front into the vacuum ($\xi' = 0$, $\eta' = 2/(\gamma-1)$) all the characteristics pass through this point. In Fig. 7 we show the pattern of the characteristics for $\gamma = 5/3$ obtained by calculation: for $N=0$ in the upper half-plane, and for $N=1$ in the lower half-plane. The point V approximates the limiting point for characteristics emerging from $G'E'$. Part of the flow front WVR' in general is curvilinear, the parts WP and VR' are

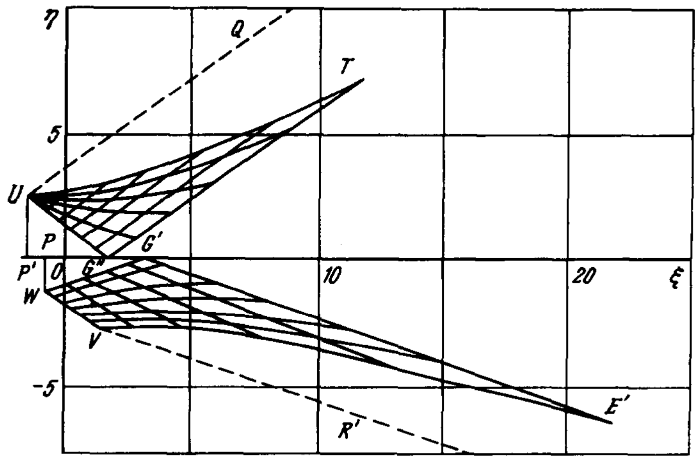

Fig. 7.

rectilinear, and W and V are points where the smoothness of the front breaks down.

Note that for unmatched α and γ it is necessary to find the integral curve of Eq. (1.10) passing through a saddle-type point, while the quantity $\lambda = \lambda_w$ corresponding to the velocity of sound becoming zero, is given by the condition

$$A(\lambda_w) - 2A^2(\lambda_w) - A'^2(\lambda_w)/2 = 0$$

The $A(\lambda)$ obtained defines the flow in the analogue of the region $WG'P$, while the solution can be obtained by the method of characteristics in the analogue of the region $R'WG'E'$. The characteristic WG' in general will not be rectilinear.

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